VARIATIONAL METHODS OF SOLUTION OF PLASTICITY PROBLEMS

(O VARIATSIONNYKH METODAKH RESHENIIA ZADACH TEORII PLASTICHNOSTI)

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1. Variational methods, based on the principle of minimum potential energy of a system and on the principle of Castinliano, have been applied extensively to the solution of problems in the theory of elasticity. In the theory of plasticity there are also analogous minimum principles; here, however, the functionals are not quadratic; therefore, the application of variational methods meets with great difficulties. Below a method is indicated which enables a successful application of variational methods to the solution of problems in theory of both plasticity and steady state creep. A similar method can also be used in finding the minimum in other nonlinear problems.

Let the body obey the equations of the theory of elastoplastic deformations

$$D_\varepsilon = \frac{1}{2} g_2(T) D_\sigma, \qquad \varepsilon = 3k\sigma \tag{1}$$

where D_ϵ, D_σ are deformation and strain deviations, ϵ the relative change of volume, σ the mean pressure, k the bulk modulus. Shear stress intensity T is connected with shear strain intensity by the relation

$$T = g_1(\Gamma)\Gamma, \qquad \text{or} \qquad \Gamma = g_2(T) T \tag{2}$$

where

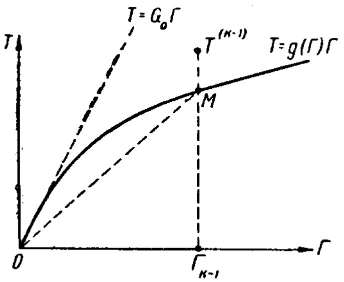
$$0 < g_1(\Gamma) \leqslant G_0, \quad g_1'(\Gamma) < 0, \qquad g_2(T) \geqslant \frac{1}{G_0}, \quad g_2'(T) > 0 \tag{3}$$

and G_0 is the shear modulus. The function $g(\Gamma)$ is equal to the tangent of the angle of slope of the secant OM (Fig. 1), and $g_2(T)$ is the cotangent of the same angle.

Let us assume, for the sake of simplicity, that spatial forces are absent. Let the load F_n be given over a portion of the surface S_F, and the displacement $\mathbf{u}$ - over the remaining portion S_u.

2. It is known [1] that the true displacement $\mathbf{u} = \mathbf{u}(x, y, z)$ corresponds to the minimum potential energy of the system

$$Э \equiv \int_V \left[\frac{\varepsilon^2}{6k} + \int g_1(\Gamma)\,\Gamma d\Gamma\right] dV - \int_{S_F} \mathbf{F}_n \cdot \mathbf{u}\; dS = \min \tag{4}$$



where V is the volume of the body (when $g_1(\Gamma)$ = const = G_0, we have the case of an elastic body). The direct calculation of its minimum by the Ritz method is difficult. Under homogeneous conditions over S_u, one can easily obtain a rough approximation in the form $\mathbf{u} = c\mathbf{u}_*$, where c is an arbitrary parameter, and $\mathbf{u}$ is a suitable displacement ($\mathbf{u}_*$ is usually taken as the solution of the corresponding elastic problem*).

Let us look for the solution $\mathbf{u}$ of the problem (4) by successive approximations in the form

$$\mathbf{u}_k = \mathbf{u}_0^* + \sum_{s=1} c_{ks}\mathbf{u}_s^* \qquad (k_1 = 0, 1, 2, \ldots) \tag{5}$$

where $\mathbf{u}_0^*$ satisfies the given conditions on S_u, $\mathbf{u}_s^*$ becomes zero on S_u and c_{ks} are arbitrary constants. In the initial (zero) approximation

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* The unreliability of this method can be judged in the case of bending of a cantilever beam by a force applied at the end $z = l$. For a power law $\epsilon_z = B\sigma_z^m (m > 1)$ the ratio of deflection $u_1(l)$ under action of force, using the approximate solution in the form mentioned, to the exact value of deflection $u(l)$ is equal to

$$\frac{u_1(l)}{u(l)} = \frac{m+2}{3}\left(\frac{2m+1}{3m}\right)^m$$

The cause of this discrepancy, increasing with m, is a considerable violation of equilibrium conditions.

(k = 0) we assume $g_1(\Gamma) = \text{const} = G_0$; having obtained $\mathbf{u}_0$, we compute the corresponding Γ_0. Using Γ_0 we compute the secant modulus $g_1(\Gamma_0)$; substituting into (4), we look for the first approximation of $\mathbf{u}_1$ by minimizing the quadratic functional:

$$\partial_1 \equiv \int_V \left[\frac{\varepsilon^2}{6k} + g_1(\Gamma_0)\frac{\Gamma^2}{2}\right] dV - \int_{S_F} \mathbf{F}_n \cdot \mathbf{u}\, dS = \min$$

etc. The presence of the variable "modulus" $g_1(\Gamma_{k-1})$ in the k-th approximation only slightly complicates the compulation of integrals; the k-th approximation has the same form as for the elastic body.

3. True stresses $\sigma_{ij}(i, j = 1, 2, 3)$ produce the minimum complementary work of the body [1]

$$R \equiv \int_V \left[\frac{3}{2} k\sigma^2 + \int g_2(T)\, T dT\right] dV = \min \qquad (6)$$

subject to the condition that

$$\int_S \delta \mathbf{F}_n \cdot \mathbf{u}\, dS = 0$$

Let us construct successive approximations in the form

$$\sigma_{ij}^{(k)} = \sigma_{ij}^{\bullet} + \sum_{s=1} c_{ks}\sigma_{ijs}^{*} \quad (k = 0, 1, 2, \ldots) \qquad (7)$$

where σ_{ij}^{0} is a particular solution of the equilibrium equations, satisfying given conditions on S_p, σ_{ijs} are particular solutions of equilibrium equations, satisfying zero boundary conditions on S_F and c_{ks} are arbitrary constants. Assuming $g_2(T) = G_0^{-1}$ we find the initial (zero) approximation $\sigma_{ij}^{(0)}$, corresponding to the elastic problem, and compute $T^{(0)}$. We assume $G_1 = g_1(T^{(0)}/G_0)$ and determine the first approximation $\sigma_{ij}^{(1)}$ by minimizing the quadratic functional:

$$R_1 \equiv \int_V \left[\frac{3}{2} k\sigma^2 + \frac{1}{G_1}\frac{T^2}{2}\right] dV = \min$$

etc. For the k-th approximation $G_k = g_1(T^{(k-1)}/G_{k-1})$. Note that the computation of the variable "modulus" G_k from the corresponding intensity of deformation Γ_{k-1} rather than from the intensity $T^{(k-1)}$ considerably improves convergence.

4. Solutions of problems in the theory of elasticity by variational methods can easily be extended to corresponding problems of plasticity with strain hardening. In (5), (7) it is advisable to retain the number of terms which ensures the desired accuracy of the solution of the elastic problem. It is easier to carry out integrations numerically; the values of the secant "modulus" G_k can be obtained starting directly from the experimental curve $T - \Gamma$. Preservation of the same form of solution in each approximation (only coefficients c_{ks} vary) considerably simplifies computations, and contrary to other methods of successive approximations (see [2]), eliminates the cumbersomeness of results obtained.

Elasto-plastic problems are solved using the same methods. Finally, note that the solution of the minimum problem in each phase of approximation can be constructed by other means (e.g. L.V. Kantorovich's method consisting of reduction the ordinary differential equations).

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